On the singularities of $3 - D$ Protter's problem for the wave equation

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ABSTRACT. In this paper we investigate some boundary value problems for the wave equation, which are three-dimensional analogues of Darboux-problems (or Cauchy-Goursat problems) on the plane. These problems have been formulated and studied by M. Protter (1954) in a $3 - D$ domain Ω_0, bounded by two characteristic cones and a plane region. Many authors studied these problems using different methods, like: Wiener-Hopf method, special Legendre functions, a priori estimates, nonlocal regularization and others. It is shown that for any $n \in N$ there exists a $C^n(\bar{\Omega}_0)$ - function, for which the corresponding unique generalized solution belongs to $C^n(\bar{\Omega}_0 \backslash O)$, but it has a strong power-type singularity at the point O. This singularity is isolated only at the vertex O of the characteristic cone and does not propagate along the cone. In this paper we investigate the exact behavior of the singular solutions at the point O. Also, we study more general boundary value problems and find that there exist infinite number of smooth right-hand side functions for which the corresponding unique generalized solutions are singular. Finally, some weight a priori estimates are stated.

1. Introduction

Consider the wave equation

$$\square u \equiv \Delta_x u - u_{tt} = \frac{1}{\varrho}(\varrho u_\varrho)_\varrho + \frac{1}{\varrho^2}u_{\varphi\varphi} - u_{tt} = f \tag{1.1}$$

in polar or Cartesian coordinates $x_1 = \varrho\cos\varphi$, $x_2 = \varrho\sin\varphi, t$ in a simply connected region $\Omega_0 \subset \mathbb{R}^3$. The region

$$\Omega_0 := \{(x_1, x_2, t) : 0 < t < 1/2, t < \sqrt{x_1^2 + x_2^2} < 1 - t\}$$

is bounded by the disk

$$\Sigma_0 := \{(x_1, x_2, t) : t = 0, x_1^2 + x_2^2 < 1\}$$

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and the characteristic surfaces of (1.1):

$$\Sigma_1 := \{(x_1, x_2, t) : 0 < t < 1/2, \sqrt{x_1^2 + x_2^2} = 1 - t\},$$
$$\Sigma_{2,0} := \{(x_1, x_2, t) : 0 < t < 1/2, \sqrt{x_1^2 + x_2^2} = t\}.$$

In this work we seek sufficient conditions for the existence and uniqueness of a generalized solution of

Problem $\mathbf{P}_\alpha$. Find a solution of the wave equation (1.1) in Ω_0, which satisfies the boundary conditions

$$P_\alpha : \quad u|_{\Sigma_1} = 0, \quad [u_t + \alpha u]|_{\Sigma_0} = 0, \tag{1.2}$$

where $\alpha \in C(\Sigma_0)$.

The adjoint problem to $\mathbf{P}_\alpha$ is

Problem $\mathbf{P}^*_\alpha$ Find a solution of the wave equation (1.1) in Ω_0 with the boundary conditions:

$$\mathbf{P}^*_\alpha : \quad u|_{\Sigma_{2,0}} = 0, \quad [u_t + \alpha u]|_{\Sigma_0} = 0 \quad . \tag{1.3}$$

The following problems, due to Protter [22], are known as

Protter's Problems. Find a solution of the wave equation (1.1) in Ω_0 with the boundary conditions

$$\begin{array}{llll} P1: & u|_{\Sigma_0 \cup \Sigma_1} = 0, & P1^*: & u|_{\Sigma_0 \cup \Sigma_{2,0}} = 0\,; \\ P2: & u|_{\Sigma_1} = 0, u_t|_{\Sigma_0} = 0, & P2^*: & u|_{\Sigma_{2,0}} = 0, u_t|_{\Sigma_0} = 0\,. \end{array} \tag{1.4}$$

The boundary conditions in problem $P1^*$ (respectively of $P2^*$) are the adjoint boundary conditions to such ones of $P1$ (respectively $P2$) for the equation (1.1) in Ω_0. Protter [22] formulated and investigated problems $P1$ and $P1^*$ in Ω_0 as multidimensional analogues of the Darboux problem on the plane. It is well known that the corresponding Darboux problems in $\mathbb{R}^2$ are well posed, but this is not true for the Protter's problems in $\mathbb{R}^3$. For recent known results concerning the problems (1.4) see papers of Popivanov, Schneider [20], [21] and references therein. For further publications in this area see: [2], [3], [7], [11], [14], [15], [18]. In [1], using Wiener-Hopf techniques for the case $\alpha(\rho) = c/\rho, c \neq 0$, Aldashev studied the Problems P_α and P^*_α. For Problem P_α, which we study in this paper, in [1] he claimed uniqueness of the solution of the class $C^1(\bar{\Omega}_0) \cap C^2(\Omega_0)$, but he did not mention any possible singular solutions.

On the other hand, Bazarbekov [5] gives another analogue of the classical Darboux problem in the same domain Ω_0. Some different statements of Darboux type problems can be found in [4], [6], [13], [16] in bounded or unbounded domains different from Ω_0.

Next, we present here the following well known (see [24], [19])

THEOREM 1.1. *For all $n \in \mathbb{N}, n \geq 4; a_n, b_n$ arbitrary constants, the functions*

$$v_n(\varrho, \varphi, t) = t\varrho^{-n}[\varrho^2 - t^2]^{n-\frac{3}{2}}(a_n \cos n\varphi + b_n \sin n\varphi) \tag{1.5}$$

are classical solutions of the homogeneous problem $P1^*$ and the functions

$$w_n(\varrho, \varphi, t) = \varrho^{-n}[\varrho^2 - t^2]^{n-\frac{1}{2}}(a_n \cos n\varphi + b_n \sin n\varphi) \tag{1.6}$$

are classical solutions of the homogeneous problem $P2^*$.

This theorem shows that for the classical solvability of the problem $P1$ (respectively, $P2$) the function f at least must be orthogonal to all functions (1.5) (respectively,(1.6)). Using Theorem 1.1, Popivanov, Schneider [**21**] proved the existence of some *generalized solutions* of Problems $P1$ and $P2$, which have at least power-type singularities at the vertex (0,0,0) of the cone $\Sigma_{2,0}$. For the homogeneous Problem $P_\alpha{}^*$ (except the case $\alpha \equiv 0$, i.e. except Problem $P2^*$) we do not know solutions analogous to (1.5) and (1.6). Anyway, in the present paper we prove results (see, Theorems 6.1 and 6.2), which ensure the existence of many singular solutions. Here we refer also to Khe Kan Cher [**15**], who gives some nontrivial solutions found for the homogeneous Problems $P1^*$ and $P2^*$, but for the Euler-Poisson-Darboux equation, which are closely connected with the results of Theorem 1.1.

In order to obtain our results, we give the following definition of a solution of Problem P_α with a possible singularity at (0,0,0).

DEFINITION 1.1. *A function $u = u(x_1,x_2,t)$ is called a generalized solution of the problem*

$$P_\alpha : \qquad \Box u = f, \qquad u\big|_{\Sigma_1} = 0, \qquad [u_t + \alpha(x)u]\big|_{\Sigma_0} = 0,$$

in Ω_0, if:

1) $u \in C^1(\bar{\Omega}_0 \setminus (0,0,0))$, $[u_t + \alpha(x)u]\big|_{\Sigma_0\setminus(0,0,0)} = 0$, $u\big|_{\Sigma_1} = 0$,

2) the identity

$$\int\limits_{\Omega_0} [u_t v_t - u_{x_1} v_{x_1} - u_{x_2} v_{x_2} - fv]dx_1 dx_2 dt = \int\limits_{\Sigma_0} \alpha(x)(uv)(x,0)dx_1 dx_2 \tag{1.7}$$

holds for all $v \in V_0 :=$

$$\{v \in C^1(\bar{\Omega}_0) : [v_t + \alpha(x)v]\big|_{\Sigma_0} = 0,\ v = 0 \text{ in a neighbourhood of } \Sigma_{2,0}\}.$$

In order to deal successfully with the encountered difficulties, as are the singularities on the cone $\Sigma_{2,0}$, we introduce the region

$$\Omega_\varepsilon = \Omega_0 \cap \{\rho - t > \varepsilon\}, \varepsilon \in [0, 1),$$

which in polar coordinates becomes

$$\Omega_\varepsilon = \{(\varrho, \varphi, t) : t > 0,\ 0 \le \varphi < 2\pi,\ \varepsilon + t < \varrho < 1 - t\}. \tag{1.8}$$

and we define the notion of a *generalized solution* of Problem P_α in $\Omega_\varepsilon, \varepsilon \in (0,1)$ (see Definition 2.1). Note that, if a generalized solution u belongs to $C^1(\bar{\Omega}_\varepsilon) \cap C^2(\Omega_\varepsilon)$, it is called a *classical solution* of Problem P_α in Ω_ε, $\varepsilon \in (0,1)$, and it satisfies the wave equation (1.1) in Ω_ε. It should be pointed out that the case $\varepsilon = 0$ is totally different from the case $\varepsilon \neq 0$.

This paper, besides Introduction, consists of five more sections. In Section 2, using some appropriate technics, we formulate the $2-D$ boundary problems $P_{\alpha,1}$ and $P_{\alpha,2}$, corresponding to the $3-D$ Problem P_α. The aim of Section 3 is to treat Problem $P_{\alpha,2}$. For this reason, we construct and study the integral equation assigned to the under consideration wave equation of general form. Also we present results concerning the classical solutions of Problem $P_{\alpha,2}$ in $\Omega_\varepsilon, \varepsilon \in (0,1)$ and give corresponding a priori estimates. In Section 4 we prove Theorems 4.1 and 4.2 which ensure the existence and uniqueness of a generalized solution of Problem $P_{\alpha,1}$ in $\Omega_\varepsilon, \varepsilon \in [0,1)$. Using the results of the previous section, in Section 5 we study the existence and uniqueness of a generalized solution of $3-D$ Problem P_α. More precisely, Theorem 5.1 ensure the uniqueness of a generalized solution of problem P_α in $\Omega_\varepsilon, \varepsilon \in [0,1)$, while Theorems 5.2 and 5.3 ensure the existence of a generalized solution, satisfying corresponding a priori estimates for problem P_α in the case, where the right-hand side of the wave equation is a trigonometric polynomial or trigonometric series. Finally, in Section 6 we present some singular generalized solutions which are smooth enough away from the point $(0,0,0)$, while at the point $(0,0,0)$ they have power-type singularity of the class ρ^{-n}. Precisely, in Theorem 6.1 we prove the following result:

Let $\alpha \geq 0$ and $\alpha \in C^\infty$. Then for each $n \in N$, $n \geq 4$, there exists a function $f_n(\varrho, \varphi, t) \in C^{n-2}(\bar{\Omega}_0)$, for which the corresponding general solution u_n of the problem P_α belongs to $C^n(\bar{\Omega}_0 \backslash (0,0,0))$ and satisfies the estimate

$$|u_n(\rho, \varphi, \rho)| \geq \rho^{-n} |\cos n\varphi|, \qquad 0 < \rho < 1. \tag{1.9}$$

When $\alpha \equiv 0$ the upper estimate holds, and in this case we have also the following two-sided estimate

$$\rho^{-n} |\cos n\varphi| \leq |u_n(\rho, \varphi, \rho)|, \qquad |u_n(\rho, \varphi, 0)| \leq C_2 \rho^{-n} |\cos n\varphi|, \tag{1.10}$$

with $C_2 = const$. That is, in the case of Problem P2 the exact behavior of $u_n(x_1, x_2, t)$ around (0,0,0) is $(x_1^2 + x_2^2)^{-n/2}$.

REMARK 1.1. *In Theorem 6.2 we find some different singular solutions for the same problem P_α. It is particularly interesting that for any parameter $\alpha(x) \geq 0$, involved in the boundary condition (1.2) on Σ_0, there are infinitely many singular solutions of the wave equation. Note, that all these solutions have strong singularities at the vertex $(0,0,0)$ of the cone $\Sigma_{2,0}$. These singularities of generalized solutions do not propagate in the direction of the bicharacteristics on the characteristic cone. It is traditionally assumed that the wave equation with right-hand side sufficiently smooth in $\bar{\Omega}_0$ cannot have a solution with an isolated singular point. For results concerning the propagation of singularities for second order operators, see Hörmander* [**10**]*, Chapter 24.5. For some related results in the case of plane Darboux-Problem, see* [**17**].

REMARK 1.2. *In 1960 Garabedian proved* [**8**] *the uniqueness of classical solution of Problem P1. Existence and uniqueness results for a generalized solution of Problems P1 and P2 can be found in* [**20**], [**21**].

REMARK 1.3. *Considering Problems P1 and P2, Popivanov, Schneider* [**19**] *announced the existence of singular solutions of both wave and degenerate hyperbolic equation. The proofs of that results are given in* [**21**] *and* [**20**] *respectively. First a priori estimates for singular solutions of Protter's Problems P1 and P2, concerning the wave equation in $\mathbb{R}^3$, were obtained in* [**21**]. *In* [**2**] *Aldashev mentions the results of* [**19**] *and, for the case of the wave equation in $\mathbb{R}^{m+1}$, he shows that there exist solutions of Problem P1 (respectively, P2) in the domain Ω_ε, which grow up on the cones $\Sigma_{2,\varepsilon}$ like $\varepsilon^{-(n+m-2)}$ (respectively, $\varepsilon^{-(n+m-1)}$), when for $\varepsilon \to 0$ the cones $\Sigma_{2,\varepsilon} := \{\rho = t + \varepsilon\}$ approximate $\Sigma_{2,0}$. It is obvious that for $m = 2$ this results can be compared with the estimate (1.10) of Theorem 6.1 and the analogous estimate of Theorem 6.2. More comments, concerning Aldashev's results* [**2**]*, we give in Section 6. Finally, we point out that in the case of an equation, which involves the wave operator and nonzero lower terms, Karatoprakliev* [**12**] *obtained a priori estimates for the smooth solutions of Problem P1 in Ω_0.*

2. Preliminaries

In this section we consider the wave equation (1.1) in a simply connected region

$$\Omega_\varepsilon := \{(\varrho, \varphi, t) : 0 < t < (1-\varepsilon)/2,\ 0 \le \varphi < 2\pi,\ \varepsilon + t < \varrho < 1 - t\}, \tag{2.1}$$

bounded by the disc $\Sigma_0 := \{(\varrho, \varphi, t) : t = 0, \varrho < 1\}$ and the characteristic surfaces of (1.1)

$$\Sigma_1 := \{(\varrho, \varphi, t) : 0 \le \varphi < 2\pi, \varrho = 1 - t\},$$
$$\Sigma_{2,\varepsilon} := \{(\varrho, \varphi, t) : 0 \le \varphi < 2\pi, \varrho = \varepsilon + t\}.$$

We seek sufficient conditions for the existence and uniqueness of a generalized solution of the equation (1.1) with $f \in C(\bar{\Omega}_\varepsilon)$, which satisfies the following boundary conditions:

$$P_\alpha : \quad u\big|_{\Sigma_1 \cap \partial\Omega_\varepsilon} = 0, \quad [u_t + \alpha(\rho)u]\big|_{\Sigma_0 \cap \partial\Omega_\varepsilon} = 0; \tag{2.2}$$

$$P^*_\alpha : \quad u\big|_{\Sigma_{2,\varepsilon}} = 0, \quad [u_t + \alpha(\rho)u]\big|_{\Sigma_0 \cap \partial\Omega_\varepsilon} = 0, \tag{2.3}$$

where for the sake of simplicity, we set $\alpha(x) \equiv \alpha(|x|) = \alpha(\rho) \in C^1([0,1])$. The problem P^*_α is the adjoint problem to Problem P_α in Ω_ε.

Now, to obtain our results we define the notion of a generalized solution as follows.

DEFINITION 2.1. *A function $u = u(\varrho, \varphi, t)$ is called a generalized solution of Problem P_α in Ω_ε, $\varepsilon > 0$, if:*
1) $u \in C^1(\bar{\Omega}_\varepsilon)$, $u\big|_{\Sigma_1 \cap \partial\Omega_\varepsilon} = 0$; $[u_t + \alpha(\varrho)u]\big|_{\Sigma_0 \cap \partial\Omega_\varepsilon} = 0$,
2) the identity

$$\int_{\Omega_\varepsilon} [u_t v_t - u_\varrho v_\varrho - \frac{1}{\varrho^2} u_\varphi v_\varphi - fv]\varrho\, d\varrho\, d\varphi\, dt = \int_{\Sigma_0 \cap \partial\Omega_\varepsilon} \varrho\alpha(\varrho) uv\, d\varrho d\varphi \tag{2.4}$$

holds for all

$$v \in V_\varepsilon := \{v \in C^1(\bar{\Omega}_\varepsilon) : [v_t + \alpha(\varrho)v]\big|_{\Sigma_0} = 0, v\big|_{\Sigma_{2,\varepsilon}} = 0\}$$

The following proposition describes the properties of generalized solutions of Problem P_α in Ω_ε.

LEMMA 2.1. *Each generalized solution of Problem P_α in Ω_0 is also a generalized solution of the same problem in Ω_ε for $\varepsilon > 0$.*

In view of (1.7), the equality (2.4) holds for each function $v \in V_0$ with the property $v \equiv 0$ in $\Omega_0 \setminus \Omega_\varepsilon$. To approximate an arbitrary function $v_1 \in V_\varepsilon$ by such functions in $W^1_2(\Omega_\varepsilon)$ we make the following steps:

Step 1. Setting $v_2(\varrho, \varphi, t) = e^{t\alpha(\varrho)} v_1(\varrho, \varphi, t)$, we get

$$\frac{\partial v_2}{\partial t}\Big|_{\Sigma_0} = 0, \qquad v_2\big|_{\Sigma_{2,\varepsilon}} = 0 \tag{2.5}$$

Step 2. The function $v_2(\varrho, \varphi, t)$ could be approximated in $W^1_2(\Omega_\varepsilon)$ by some functions, which satisfy (2.5) and are zero in a neighborhood of the circle

$\{\varrho = \varepsilon, t = 0\}$. In fact, such functions are:

$$v_{2m}(\varrho,\varphi,t) := v_2(\varrho,\varphi,t)\psi(m\sqrt{(\varrho-\varepsilon)^2+t^2}), \quad m \to \infty,$$

where $\psi \in C^\infty(R^1)$, $\psi(s) = 0$, for $s \le 1$ and $\psi(s) = 1$, for $s > 2$.

Step 3. Each function $v_{2m}(\varrho,\varphi,t)$ could be approximated in $W_2^1(\Omega_\varepsilon)$ by some functions, which satisfy (2.5), and are zero in a neighborhood of the cone $\{\varrho = t + \varepsilon\}$:

$$v_k(\varrho,\varphi,t) := v_{2m}(\varrho,\varphi,t)\psi((t-\varrho+\varepsilon)k), \quad k \to \infty.$$

In the special, but main case, when

$$f(\varrho,\varphi,t) = f_n^{(1)}(\varrho,t)\cos n\varphi + f_n^{(2)}(\varrho,t)\sin n\varphi \tag{2.6}$$

we ask the generalized solution to be of the form

$$u(\varrho,\varphi,t) = u_n^{(1)}(\varrho,t)\cos n\varphi + u_n^{(2)}(\varrho,t)\sin n\varphi \tag{2.7}$$

If we introduce the function $u^{(1)}(\varrho,t) := \begin{cases} u_n^{(1)} & \text{for } f^{(1)} = f_n^{(1)}, \\ u_n^{(2)} & \text{for } f^{(1)} = f_n^{(2)}, \end{cases}$

then, in view of (1.1), we conclude that

$$\tilde{\Box} u^{(1)} = \frac{1}{\varrho}(\varrho u_\varrho^{(1)})_\varrho - \frac{n^2}{\varrho^2}u^{(1)} - u_{tt}^{(1)} = f^{(1)} \tag{2.8}$$

in $G_\varepsilon = \{(\varrho,t) : t > 0, \varepsilon + t < \varrho < 1 - t\}$, which is bounded by the sets:

$$\begin{aligned} &S_0 = \{(\varrho,t) : t = 0, 0 < \varrho < 1\}, \\ &S_1 = \{(\varrho,t) : \varrho = 1 - t\}, \quad S_{2,\varepsilon} = \{(\varrho,t) : \varrho = t + \varepsilon\}. \end{aligned} \tag{2.9}$$

Instead of the equation (2.8), consider the more general equation

$$Lu^{(1)} = \frac{1}{\varrho}(\varrho u_\varrho^{(1)})_\varrho - u_{tt}^{(1)} + d(\rho,t)u^{(1)} = f^{(1)}, \tag{2.10}$$

with the same boundary conditions. In this case, the two-dimensional problem corresponding to P_α is

$$P_{\alpha,1}: \quad \begin{cases} Lu^{(1)} = f^{(1)} \text{ in } G_\varepsilon, \\ u^{(1)}\big|_{S_1} = 0, \qquad [u_t^{(1)} + \alpha(\varrho)u^{(1)}]\big|_{S_0\setminus(0,0)} = 0 \end{cases} \tag{2.11}$$

and its generalized solution is defined by

Definition 2.2. *A function $u^{(1)} = u^{(1)}(\varrho,t)$ is called a generalized solution of problem $P_{\alpha,1}$ in G_ε, $\varepsilon > 0$, if:*

1) $u \in C^1(\bar{G}_\varepsilon)$, $[u_t + \alpha(\varrho)u]\big|_{S_0\cap\partial G_\varepsilon} = 0$, $u\big|_{S_1\cap\partial G_\varepsilon} = 0$;

2) the identity

$$\int_{G_\varepsilon}[u_t^{(1)}v_t - u_\varrho^{(1)}v_\varrho + d(\varrho,t)u^{(1)}v - f^{(1)}v]\varrho d\varrho\, dt = \int_{S_0\cap\partial G_\varepsilon} \varrho\alpha(\varrho)u^{(1)}v\, d\varrho \tag{2.12}$$

holds for all

$$v \in V_\varepsilon^{(1)} = \{v \in C^1(\bar{G}_\varepsilon) : [v_t + \alpha(\rho)v]\big|_{S_0} = 0, v\big|_{S_{2,\varepsilon}} = 0\}.$$

By introducing a new function

$$u^{(2)}(\varrho,t)=\varrho^{\frac{1}{2}}u^{(1)}(\varrho,t), \tag{2.13}$$

we transform (2.10) to the equation

$$u^{(2)}_{\varrho\varrho}-u^{(2)}_{tt}+\left[d(\varrho,t)+\frac{1}{4\varrho^2}\right]u^{(2)}=\varrho^{\frac{1}{2}}f^{(1)}, \tag{2.14}$$

with the string operator in the main part. Substituting the new coordinates

$$\xi=1-\varrho-t,\eta=1-\varrho+t, \tag{2.15}$$

from (2.14) we derive

$$U_{\xi\eta}+\frac{1}{4}\left[d^{(2)}(\xi,\eta)+(2-\eta-\xi)^{-2}\right]U=\frac{1}{4\sqrt{2}}(2-\eta-\xi)^{\frac{1}{2}}F(\xi,\eta), \tag{2.16}$$

in $D_\varepsilon=\{(\xi,\eta):0<\xi<\eta<1-\varepsilon\}$, where

$$U(\xi,\eta)=u^{(2)}(\rho(\xi,\eta),t(\xi,\eta)),\qquad F(\xi,\eta)=f^{(1)}(\rho(\xi,\eta),t(\xi,\eta)). \tag{2.17}$$

Thus, we reduced the problem $P_{\alpha,1}$ to the Darboux-Goursat problem for the more general equation (2.10) with the same boundary conditions:

$$P_{\alpha,2}:\ \begin{cases} U_{\xi\eta}+c(\xi,\eta)U=g(\xi,\eta) \text{ in } D_\varepsilon,\\ U(0,\eta)=0,(U_\eta-U_\xi)(\xi,\xi)+\alpha(1-\xi)U(\xi,\xi)=0.\end{cases} \tag{2.18}$$

In view of the above observations, the wave equation (1.1) transforms finally to the equation

$$U_{\xi\eta}+\frac{1-4n^2}{4(2-\xi-\eta)^2}U=\frac{1}{4\sqrt{2}}(2-\eta-\xi)^{\frac{1}{2}}F(\xi,\eta), \tag{2.19}$$

which is of the form (2.16).

3. The integral equation corresponding to Problem $P_{\alpha,2}$

Set

$$\begin{aligned} c(\xi,\eta)&=\frac{1-4n^2}{4(2-\xi-\eta)^2}\in C^\infty(\bar{D}_0\setminus(1,1)),\\ g(\xi,\eta)&=\frac{1}{4\sqrt{2}}(2-\xi-\eta)^{\frac{1}{2}}F(\xi,\eta).\end{aligned} \tag{3.1}$$

Then the equation (2.19), in new therms, takes the form of the equation in (2.18). Remark, that if $f_n^{(i)}\in C^0(\bar{G}_0)$, $i=1,2,$ then $g\in C(\bar{D}_0)$, while $f_n^{(i)}\in C^k(\bar{G}_0)$, $i=1,2$, then $g\in C^k(\bar{D}_0\backslash(1,1))$.

In order to investigate the smoothness and the singularity of a solution of the original $3-D$ problem P_α on $\Sigma_{2,0}$, we are seeking for a classical solution of the corresponding $2-D$ problem $P_{\alpha,2}$ not only in the domain D_ε, but also in the domain

$$D^{(1)}_\varepsilon:=\{(\xi,\eta):0<\xi<\eta<1,0<\xi<1-\varepsilon\},\quad \varepsilon>0. \tag{3.2}$$

Clearly, $D_\varepsilon\subset D^{(1)}_\varepsilon$.

Consider now the equation from (2.18), i.e.

$$U_{\xi\eta}+c(\xi,\eta)U=g(\xi,\eta) \text{ in } D^{(1)}_\varepsilon, \tag{3.3}$$

where $c(\xi,\eta)\in C(\bar{D}^{(1)}_\varepsilon)$, $g(\xi,\eta)\in C(\bar{D}^{(1)}_\varepsilon)$, $\varepsilon>0$.

Next, for any $(\xi_0, \eta_0) \in D_\varepsilon^{(1)}$, we consider the sets

$$\Pi := \{(\xi, \eta) : 0 < \xi < \xi_0, \xi_0 < \eta < \eta_0\}, \quad T := \{(\xi, \eta) : 0 < \xi < \eta, 0 < \eta < \xi_0\}$$

and we construct an equivalent integral equation to the problem $P_{\alpha,2}$, in such a way that any solution of the problem $P_{\alpha,2}$ to be also a solution of the constructed integral equation. For this reason, we consider the following integrals:

$$\begin{aligned} I_0 &:= \iint_{\Pi} [g(\xi, \eta) - c(\xi, \eta)U(\xi, \eta)]\, d\eta\, d\xi = \int_0^{\xi_0} \int_{\xi_0}^{\eta_0} U_{\xi\eta}(\xi, \eta)\, d\eta\, d\xi \\ &= \int_0^{\xi_0} [U_\xi(\xi, \eta_0) - U_\xi(\xi, \xi_0)]\, d\xi = U(\xi_0, \eta_0) - U(\xi_0, \xi_0) \end{aligned}$$

and

$$\begin{aligned} I_1 &:= \iint_{T} [g(\xi, \eta) - c(\xi, \eta)U(\xi, \eta)]\, d\eta\, d\xi = \int_0^{\xi_0} \int_{\xi}^{\xi_0} U_{\xi\eta}(\xi, \eta)\, d\eta\, d\xi \\ &= \int_0^{\xi_0} [U_\xi(\xi, \xi_0) - U_\xi(\xi, \xi)]\, d\xi = U(\xi_0, \xi_0) - \int_0^{\xi_0} U_\xi(\xi, \xi)\, d\xi. \end{aligned}$$

On the other side,

$$I_1 = \int_0^{\xi_0} \int_0^{\eta} U_{\xi\eta}(\xi, \eta)\, d\xi\, d\eta = \int_0^{\xi_0} U_\eta(\eta, \eta)\, d\eta.$$

Hence, we see that:

$$\begin{aligned} 2I_1 &= U(\xi_0, \xi_0) + \int_0^{\xi_0} [U_\eta(\xi, \xi) - U_\xi(\xi, \xi)]\, d\xi \\ &= U(\xi_0, \xi_0) - \int_0^{\xi_0} \alpha(1-\xi)U(\xi, \xi)\, d\xi, \\ I_0 + 2I_1 &= U(\xi_0, \eta_0) - \int_0^{\xi_0} \alpha(1-\xi)U(\xi, \xi)\, d\xi. \end{aligned}$$

From the latest relation we obtain

$$\begin{aligned} U(\xi_0, \eta_0) &= \int_0^{\xi_0} \int_{\xi_0}^{\eta_0} [g(\xi, \eta) - c(\xi, \eta)U(\xi, \eta)]\, d\eta\, d\xi \\ &\quad + 2\int_0^{\xi_0} \int_0^{\eta} [g(\xi, \eta) - c(\xi, \eta)U(\xi, \eta)]\, d\xi\, d\eta \\ &\quad + \int_0^{\xi_0} \alpha(1-\xi)U(\xi, \xi)\, d\xi, \text{ for } (\xi_0, \eta_0) \in \bar{D}_\varepsilon^{(1)}, \end{aligned} \tag{3.4}$$

which is the desired integral equation.

Next, we set

$$M_g := \sup_{D_\varepsilon^{(1)}} |g(\xi, \eta)|, \quad c(\varepsilon) := \sup_{D_\varepsilon^{(1)}} |c(\xi, \eta)|, \quad M_\alpha := \sup_{[0,1]} |\alpha(\xi)| \tag{3.5}$$

and state the following

THEOREM 3.1. *Let $c(\xi,\eta) \in C(\bar{D}_\varepsilon^{(1)})$, $g(\xi,\eta) \in C(\bar{D}_\varepsilon^{(1)})$, $\varepsilon > 0$. Then there exists a classical solution $U(\xi,\eta) \in C^1(\bar{D}_\varepsilon^{(1)})$ of the equation* (3.3) *which satisfies the boundary conditions* (2.18) *with $U_{\xi\eta}(\xi,\eta) \in C(\bar{D}_\varepsilon^{(1)})$ and*

$$\text{(3.6)}\qquad \begin{aligned} &|U(\xi_0,\eta_0)| \le \xi_0 M_g[c(\varepsilon)+M_\alpha]^{-1}\exp[c(\varepsilon)+M_\alpha] \quad \text{in } D_\varepsilon^{(1)},\\ &\sup_{D_\varepsilon^{(1)}}\{|U_\xi|,|U_\eta|\} \le M_g[c(\varepsilon)+M_\alpha]^{-1}\exp[c(\varepsilon)+2M_\alpha]. \end{aligned}$$

Proof. In order to solve the integral equation (3.4), we use the following sequence of successive approximations $U^{(n)}$, defined by the formula

$$\text{(3.7)}\qquad \begin{aligned} U^{(n+1)}(\xi_0,\eta_0) = &\int_0^{\xi_0}\int_{\xi_0}^{\eta_0}[g(\xi,\eta)-c(\xi,\eta)U^{(n)}(\xi,\eta)]\,d\eta\,d\xi\\ &+2\int_0^{\xi_0}\int_0^{\eta}[g(\xi,\eta)-c(\xi,\eta)U^{(n)}(\xi,\eta)]\,d\xi\,d\eta\\ &+\int_0^{\xi_0}\alpha(1-\xi)U^{(n)}(\xi,\xi)\,d\xi,\\ U^{(0)}(\xi_0,\eta_0) = 0,& \quad \text{in } D_\varepsilon^1. \end{aligned}$$

We will show that for any $(\xi_0,\eta_0) \in \bar{D}_\varepsilon^{(1)}$ and $n \in \mathbb{N}$ it holds

$$\text{(3.8)}\qquad |(U^{(n+1)}-U^{(n)})(\xi_0,\eta_0)| \le \frac{M_g[c(\varepsilon)+M_\alpha]^n\xi_0^{n+1}}{(n+1)!}.$$

Indeed:

$$1)\quad U^{(1)}(\xi_0,\eta_0) = \int_0^{\xi_0}\int_{\xi_0}^{\eta_0} g(\xi,\eta)\,d\eta\,d\xi + 2\int_0^{\xi_0}\int_0^{\eta} g(\xi,\eta)\,d\xi\,d\eta,$$

and hence

$$|U^{(1)}(\xi_0,\eta_0)| \le M_g[\xi_0(\eta_0-\xi_0)+\xi_0^2] = M_g\xi_0\eta_0 \le M_g\xi_0.$$

2) Let, by the induction hypothesis (3.8),

$$|(U^{(n)}-U^{(n-1)})(\xi_0,\eta_0)| \le \frac{M_g}{n!}[c(\varepsilon)+M_\alpha]^{n-1}\xi_0^n := A_n\xi_0^n$$

be satisfied. Then, it follows that

$$\begin{aligned} |(U^{(n+1)}-U^{(n)})(\xi_0,\eta_0)| &= \Big| -\int_0^{\xi_0}\int_{\xi_0}^{\eta_0} c(\xi,\eta)(U^{(n)}-U^{(n-1)})(\xi,\eta)\,d\eta\,d\xi\\ -2\int_0^{\xi_0}\int_0^{\eta} c(\xi,\eta)&(U^{(n)}-U^{(n-1)})(\xi,\eta)\,d\xi\,d\eta + \int_0^{\xi_0}\alpha(1-\xi)(U^{(n)}-U^{(n-1)})(\xi,\xi)\,d\xi\Big|\\ &\le A_n\left[c(\varepsilon)\left(\int_0^{\xi_0}\int_{\xi_0}^{\eta_0}\xi^n\,d\eta\,d\xi + 2\int_0^{\xi_0}\int_0^{\eta}\xi^n\,d\xi\,d\eta\right) + M_\alpha\int_0^{\xi_0}\xi^n\,d\xi\right]\\ &= A_n\left[c(\varepsilon)\left(\frac{1}{n+1}\xi_0^{n+1}(\eta_0-\xi_0) + \frac{2}{(n+1)(n+2)}\xi_0^{n+2}\right) + \frac{M_\alpha}{n+1}\xi_0^{n+1}\right] \end{aligned}$$

$$= A_n\left[c(\varepsilon)\left(\frac{1}{n+1}\xi_0^{n+1}\eta_0 - \frac{n}{(n+1)(n+2)}\xi_0^{n+2}\right) + \frac{M_\alpha}{n+1}\xi_0^{n+1}\right]$$

$$\le A_n\left[\frac{c(\varepsilon)}{n+1}\xi_0^{n+1} + \frac{M_\alpha}{n+1}\xi_0^{n+1}\right] = \frac{M_g}{(n+1)!}[c(\varepsilon)+M_\alpha]^n\xi_0^{n+1} = A_{n+1}\xi_0^{n+1}.$$

So, the inequality (3.8) is proved and hence the uniform convergence of the sequence $\{U^{(m)}(\xi,\eta)\}_{m\in\mathbb{N}}$ in $\bar{D}_\varepsilon^{(1)}$ is obvious. For the limit function $U \in C(\bar{D}_\varepsilon^{(1)})$ we obtain the integral equality (3.4) and $U(0,\eta_0)=0$.

Also, in view of (3.8), we see that

$$|(U^{(n+1)}(\xi_0,\eta_0)| = \left|\sum_{k=0}^{n}(U^{(k+1)} - U^{(k)})(\xi_0,\eta_0)\right| \le \xi_0 M_g \sum_{k=0}^{n}\frac{[c(\varepsilon)+M_\alpha]^k}{(k+1)!}\xi_0^k$$

$$\le \xi_0 M_g[c(\varepsilon)+M_\alpha]^{-1}\exp[c(\varepsilon)+M_\alpha],$$

and therefore

$$|U(\xi_0,\eta_0)| \le \xi_0 M_g[c(\varepsilon)+M_\alpha]^{-1}\exp[c(\varepsilon)+M_\alpha].$$

To estimate the first derivatives of the function U, by (3.7), we get:

$$U_{\xi_0}^{(n+1)}(\xi_0,\eta_0) = \alpha(1-\xi_0)U^{(n)}(\xi_0,\xi_0) \tag{3.9}$$

$$+\int_0^{\xi_0}[g(\xi,\xi_0) - c(\xi,\xi_0)U^{(n)}(\xi,\xi_0)]\,d\xi + \int_{\xi_0}^{\eta_0}[g(\xi_0,\eta) - c(\xi_0,\eta)U^{(n)}(\xi_0,\eta)]\,d\eta,$$

and

$$U_{\eta_0}^{(n+1)}(\xi_0,\eta_0) = \int_0^{\xi_0}[g(\xi,\eta_0) - c(\xi,\eta_0)U^{(n)}(\xi,\eta_0)]\,d\xi. \tag{3.10}$$

Using (3.8) and (3.9) we see that

$$|U_{\xi_0}^{(1)}(\xi_0,\eta_0)| = \left|\int_0^{\xi_0} g(\xi,\xi_0)\,d\xi + \int_{\xi_0}^{\eta_0} g(\xi_0,\eta)\,d\eta\right|$$

$$\le M_g(\xi_0+\eta_0-\xi_0) = M_g\eta_0 \le M_g,$$

and

$$|(U_{\xi_0}^{(n+1)} - U_{\xi_0}^{(n)})(\xi_0,\eta_0)| = \Big| -\int_0^{\xi_0} c(\xi,\xi_0)(U^{(n)} - U^{(n-1)})(\xi,\xi_0)]\,d\xi$$

$$-\int_{\xi_0}^{\eta_0} c(\xi_0,\eta)(U^{(n)} - U^{(n-1)})(\xi_0,\eta)\,d\eta + \alpha(1-\xi_0)(U^{(n)} - U^{(n-1)})(\xi_0,\xi_0)\Big|$$

$$\le \frac{M_g}{n!}[c(\varepsilon)+M_\alpha]^{n-1}\left[c(\varepsilon)\left(\int_0^{\xi_0}\xi^n\,d\xi + \int_{\xi_0}^{\eta_0}\xi_0^n\,d\eta\right) + M_\alpha\xi_0^n\right]$$

$$\le \frac{M_g}{n!}[c(\varepsilon)+M_\alpha]^{n-1}\left[\frac{c(\varepsilon)}{n+1} + M_\alpha\right].$$

So, for the derivative $U_{\xi_0}(\xi_0, \eta_0)$ we get the estimation:

$$(3.11) \qquad |U_{\xi_0}(\xi_0,\eta_0)| = |\lim U^{(n+1)}_{\xi_0}(\xi_0,\eta_0)| = |\sum_{k=0}^{\infty}(U^{(k+1)}_{\xi_0} - U^{(k)}_{\xi_0})(\xi_0,\eta_0)|$$

$$\leq M_g \sum_{k=0}^{\infty} \frac{[c(\varepsilon)+M_\alpha]^{k-1}}{k!}\left[\frac{c(\varepsilon)}{k+1} + M_\alpha\right] \leq M_g[c(\varepsilon)+M_\alpha]^{-1}\exp[c(\varepsilon)+2M_\alpha].$$

Using (3.8) and (3.10), we find

$$|(U^{(n+1)}_{\eta_0} - U^{(n)}_{\eta_0})(\xi_0,\eta_0)| = |-\int_0^{\xi_0} c(\xi,\eta_0)(U^{(n)} - U^{(n-1)})(\xi,\eta_0)]\, d\xi|$$

$$\leq \frac{c(\varepsilon)M_g}{n!}[c(\varepsilon)+M_\alpha]^{n-1}\int_0^{\xi_0}\xi^n\, d\xi \leq \frac{M_g}{(n+1)!}[c(\varepsilon)+M_\alpha]^n \xi_0^{n+1}.$$

Therefore, $U \in C^1(\bar{D}^{(1)}_\varepsilon)$ and

$$(3.12) \qquad |U_{\eta_0}(\xi_0,\eta_0)| \leq \xi_0[c(\varepsilon)+M_\alpha]^{-1}\exp[c(\varepsilon)+M_\alpha].$$

Also, by (3.10), it follows that

$$U^{(n+1)}_{\eta_0\xi_0}(\xi_0,\eta_0) = g(\xi_0,\eta_0) - c(\xi_0,\eta_0)U^{(n)}(\xi_0,\eta_0).$$

Thus, the function $U(\xi_0,\eta_0)$ is a solution of (3.3) and $U_{\xi\eta} \in C(\bar{D}^{(1)}_\varepsilon)$. Finally, using (3.9) and (3.10), we see that

$$\lim_{n\to\infty}[U^{(n+1)}_{\eta_0} - U^{(n+1)}_{\xi_0} + \alpha(1-\xi_0)U^{(n+1)}](\xi_0,\eta_0)$$

$$= \alpha(1-\xi_0)\lim_{n\to\infty}[(U^{(n+1)} - U^{(n)})(\xi_0,\xi_0)] = 0,$$

i.e. $U(\xi_0,\eta_0)$ satisfies boundary conditions (2.18). ■

The next result is very important for the investigation of the singularity of a generalized solution of problem P_α.

LEMMA 3.1. *Let $c(\xi,\eta)$, $g(\xi,\eta) \in C(\bar{D}^{(1)}_\varepsilon)$ and*

$$(3.13) \qquad g(\xi,\eta) \geq 0, \quad c(\xi,\eta) \leq 0 \quad in\ \bar{D}^{(1)}_\varepsilon;\ \alpha(\xi) \geq 0 \quad for\ 0 \leq \xi \leq 1.$$

Then for the solution $U(\xi,\eta)$ of the problem (3.3), (2.18) *(already found in Theorem 3.1) we have*

$$(3.14) \qquad U(\xi,\eta) \geq 0, \quad U_\eta(\xi,\eta) \geq 0, \quad U_\xi(\xi,\eta) \geq 0 \quad in\ \bar{D}^{(1)}_\varepsilon.$$

Proof. In view of (3.7), from (3.13) we have

$$U^{(1)}(\xi_0,\eta_0) = \int_0^{\xi_0}\int_{\xi_0}^{\eta_0} g(\xi,\eta)\, d\eta\, d\xi + 2\int_0^{\xi_0}\int_0^{\eta} g(\xi,\eta)\, d\xi\, d\eta \geq 0.$$

Suppose that $(U^{(n)} - U^{(n-1)})(\xi_0, \eta_0) \geq 0$ for some $n \in \mathbb{N}$. Then

$$(U^{(n+1)} - U^{(n)})(\xi_0, \eta_0) = -\int_0^{\xi_0}\int_{\xi_0}^{\eta_0} c(\xi,\eta)(U^{(n)} - U^{(n-1)})(\xi,\eta)\, d\eta\, d\xi$$

$$-2\int_0^{\xi_0}\int_0^{\eta} c(\xi,\eta)(U^{(n)} - U^{(n-1)})(\xi,\eta)\, d\xi\, d\eta$$

$$+\int_0^{\xi_0} \alpha(1-\xi)(U^{(n)} - U^{(n-1)})(\xi,\xi)\, d\xi \geq 0$$

and

$$U(\xi_0,\eta_0) = \sum_{n=0}^{\infty}(U^{(n+1)} - U^{(n)})(\xi_0,\eta_0) \geq 0. \tag{3.15}$$

Since $U(\xi_0,\eta_0) \geq 0$ for any $(\xi_0,\eta_0) \in \bar{D}_\varepsilon^{(1)}$ and

$$U_{\xi_0}(\xi_0,\eta_0) = \alpha(1-\xi_0)U(\xi_0,\xi_0) \tag{3.16}$$

$$+\int_0^{\xi_0} [g(\xi,\xi_0) - c(\xi,\xi_0)U(\xi,\xi_0)]\, d\xi + \int_{\xi_0}^{\eta_0} [g(\xi_0,\eta) - c(\xi_0,\eta)U(\xi_0,\eta)]\, d\eta,$$

$$U_{\eta_0}(\xi_0,\eta_0) = \int_0^{\xi_0} [g(\xi,\eta_0) - c(\xi,\eta_0)U(\xi,\eta_0)]\, d\xi, \tag{3.17}$$

we conclude that $U_{\xi_0} \geq 0$ and $U_{\eta_0} \geq 0$ in $\bar{D}_\varepsilon^{(1)}$. ∎

As an immediate consequence of Theorem 3.1, (3.16) and (3.17), we have the following

THEOREM 3.2. *Let $c(\xi,\eta) \in C^k(\bar{D}_\varepsilon^{(1)})$, $g(\xi,\eta) \in C^k(\bar{D}_\varepsilon^{(1)})$, $\alpha \in C^k((0,1])$, where $k \geq 1, \varepsilon > 0$. Then there exists a classical solution $U \in C^{k+1}(\bar{D}_\varepsilon^{(1)})$ of the problem $P_{\alpha,2}$.*

4. Existence and uniqueness theorems for $2-D$ Problem $\mathbf{P}_{\alpha,1}$

Consider the problem

$$P_{\alpha,1}: \begin{cases} Lu^{(1)} = \frac{1}{\rho}(\rho u_\rho^{(1)}) - u_{tt}^{(1)} + d(\rho,t)u^{(1)} = f^{(1)} \text{ in } G_\varepsilon, \\ u^{(1)}\big|_{S_1} = 0, \qquad [u_t^{(1)} + \alpha(\rho)u^{(1)}]\big|_{S_0} = 0. \end{cases} \tag{4.1}$$

Note that, the notion of the generalized solution of the problem $P_{\alpha,1}$ in the domain G_ε , $\varepsilon \in (0,1)$, has been defined by Definition 2.2.

THEOREM 4.1. *If $d(\rho,t), f^{(1)}(\varrho,t) \in C^1(\bar{G}_0 \setminus (0,0))$, then there exists a generalized solution $u^{(1)} \in C^2(\bar{G}_0 \setminus (0,0))$ of problem $P_{\alpha,1}$ in G_0, which is a classical solution of the problem $P_{\alpha,1}$ in any domain G_ε, $\varepsilon \in (0,1)$.*

Proof. In view of (2.13) and (2.15), i.e. $u^{(2)}(\varrho,t) = \varrho^{1/2}u^{(1)}(\varrho,t)$ and $\xi = 1-\varrho-t$, $\eta = 1-\varrho+t$, consider the function

$$U(\xi,\eta)=u^{(2)}(\varrho(\xi,\eta),t(\xi,\eta)).$$

Then Problem $P_{\alpha,1}$ (see (4.1)) becomes $P_{\alpha,2}$, i.e.

$$U_{\xi\eta} + \frac{1}{4}\left[d^{(2)}(\xi,\eta) + (2-\xi-\eta)^{-2}\right]U = \frac{1}{4\sqrt{2}}(2-\eta-\xi)^{1/2}F(\xi,\eta), \tag{4.2}$$

$$U(0,\eta) = 0, \qquad (U_\eta - U_\xi)(\xi,\xi) + \alpha(1-\xi)U(\xi,\xi) = 0. \tag{4.3}$$

For each $\varepsilon \in (0,1)$ Theorem 3.2 ensures the existence of a classical solution $U(\xi,\eta) \in C^2(\bar{D}^{(1)}_\varepsilon)$ of the problem $P_{\alpha,2}$. The inverse transformations lead to a function $u^{(1)}(\varrho,t) \in C^2(\bar{G}_0 \setminus (0,0))$, which is a classical solution of Problem $P_{\alpha,1}$ in G_ε. This solution is also a generalized solution of the same problem in G_0, because each one of test functions $v \in V_0$ is zero in $G_0 \backslash G_\varepsilon$ for some $\varepsilon > 0$ and, for the concrete v, (1.6) coincides with (2.4).

The proof of the theorem is complete. ■

THEOREM 4.2. *For each fixed $\varepsilon \in (0,1)$ there exists at most one generalized solution of the problem $P_{\alpha,1}$ in G_ε.*

Proof. If u_1 and u_2 are two generalized solutions of $P_{\alpha,1}$, then for $u^{(1)} := u_1 - u_2$ we see that

$$u^{(1)} \in C^1(\bar{G}_\varepsilon), \quad u^{(1)}|_{S_1 \cap \bar{G}_\varepsilon} = 0, \quad [u^{(1)}_t + \alpha(r)u^{(1)}]|_{S_0 \cap \bar{G}_\varepsilon} = 0$$

and the identity

$$\int_{G_\varepsilon} [u^{(1)}_t v_t - u^{(1)}_\varrho v_\varrho + d(\varrho,t)u^{(1)}v]\varrho d\varrho dt - \int_{S_0 \cap \partial G_\varepsilon} \varrho\alpha(\varrho)u^{(1)}v\, d\varrho = 0 \tag{4.4}$$

holds for all functions $v \in V^{(1)}_\varepsilon$.

Let $h(\rho,t) \in C^1(\bar{G}_0 \setminus (0,0))$. Set

$$\begin{gathered} g(\xi,\eta) := \tfrac{1}{4\sqrt{2}}[2-\xi-\eta]^{1/2}h((2-\xi-\eta)/2, (\eta-\xi)/2) \in C^1(\bar{D}^{(1)}_\varepsilon), \\ c(\xi,\eta) = \tfrac{1}{4}\left[d(\rho(\xi,\eta), t(\xi,\eta)) + (2-\eta-\xi)^{-2}\right] \in C^1(\bar{D}^{(1)}_\varepsilon), \end{gathered} \tag{4.5}$$

and consider the boundary value problem

$$V_{\xi\eta} + c(\xi,\eta)V = g(\xi,\eta) \quad \text{in } D_\varepsilon, \tag{4.6}$$

$$V\big|_{\eta=1-\varepsilon} = 0, \quad [V_\eta - V_\xi + \alpha(1-\xi)V]\big|_{\eta=\xi} = 0. \tag{4.7}$$

By using the substitutions $\xi_1 = 1-\varepsilon-\eta$, $\eta_1 = 1-\varepsilon-\xi$, and by setting

$$V^{(1)}(\xi_1,\eta_1) = V(1-\varepsilon-\eta_1, 1-\varepsilon-\xi_1), \tag{4.8}$$

the problem (4.6), (4.7) becomes

$$V^{(1)}_{\xi_1\eta_1} + c^{(1)}(\xi_1,\eta_1)V^{(1)} = g^{(1)}(\xi_1,\eta_1) \quad \text{in } D_\varepsilon, \tag{4.9}$$

$$V^{(1)}\big|_{\xi_1=0} = 0, \quad [V^{(1)}_{\eta_1} - V^{(1)}_{\xi_1} + \alpha(\varepsilon+\xi_1)V^{(1)}]\big|_{\eta_1=\xi_1} = 0 \tag{4.10}$$

where

$$c^{(1)}(\xi_1,\eta_1) = \frac{1}{4}\left[d^{(1)}(\xi_1,\eta_1) + (\xi_1+\eta_1+2\varepsilon)^{-2}\right] \in C^1(\bar{D}_\varepsilon).$$

But (4.9), (4.10) is the Goursat–Darboux problem $P_{\alpha,2}$ in the domain D_ε, for which Theorem 3.2 holds. Consequently, there exists a classical solution $V^{(1)}(\xi_1,\eta_1) \in$

C^2 of (4.9), (4.10). The inverse transformation leads to a classical solution $V = V(\xi,\eta)$ of (4.6), (4.7) in D_ε. Similar arguments show that $v(\varrho,t) = \varrho^{-1/2}V(\xi(\varrho,t),\eta(\varrho,t))$ is a classical solution of the problem

$$Lv = \frac{1}{\varrho}(\varrho v_\varrho)_\varrho - v_{tt} + dv = h(\varrho,t) \quad \text{in } G_\varepsilon, \tag{4.11}$$

$$v|_{S_{2,\varepsilon}} = 0, \qquad [v_t + \alpha(\varrho)v]|_{S_0} = 0, \tag{4.12}$$

for fixed $\varepsilon \in (0,1)$.

Multiplying (4.11) by a generalized solution $u^{(1)} \in C^1(\bar{G}_\varepsilon)$ and integrating by parts, we find

$$\int_{G_\varepsilon} [v_t u_t^{(1)} - v_\varrho u_\varrho^{(1)} + dvu^{(1)} - hu^{(1)}]\varrho\, d\varrho\, dt - \int_{S_0\cap\partial G_\varepsilon} \varrho\alpha(\varrho)vu^{(1)}\, d\varrho = 0. \tag{4.13}$$

Comparing (4.13) and (4.4), we see that

$$\int_{G_\varepsilon} h(\varrho,t)u^{(1)}(\varrho,t)\, \rho d\varrho\, dt = 0. \tag{4.14}$$

But the function $h(\varrho,t) \in C^1(\bar{G}_0 \setminus (0,0))$ has been arbitrarily chosen. Thus (4.14) gives $u^{(1)}(\varrho,t) = 0$ in G_ε. The proof is complete. ■

5. Existence and uniqueness theorems for $3-D$ Problem P_α

In this section we consider for the wave equation

$$\Box u := \frac{1}{\varrho}(\varrho u_\varrho)_\varrho + \frac{1}{\varrho^2}u_{\varphi\varphi} - u_{tt} = f(\varrho,\varphi,t), \tag{5.1}$$

subject to the following boundary value problem

$$P_\alpha: \quad \Box u = f \text{ in } \Omega_\varepsilon, \quad u|_{\Sigma_1\cap\partial\Omega_\varepsilon} = 0, \quad [u_t + \alpha(\varrho)u]|_{\Sigma_0\cap\partial\Omega_\varepsilon} = 0. \tag{5.2}$$

and prove the following results.

THEOREM 5.1. *For $0 \le \varepsilon < 1$ there exists at most one generalized solution of Problem P_α in Ω_ε.*

Proof. Case $0 < \varepsilon < 1$. If u_1, u_2 are two generalized solutions of P_α in Ω_ε, then for $u^{(1)} := u_1 - u_2 \in C^1(\bar{\Omega}_\varepsilon)$ we know that

$$u^{(1)}|_{\Sigma_1\cap\partial\Omega_\varepsilon} = 0, \qquad [u_t^{(1)} + \alpha(\varrho)u^{(1)}]|_{\Sigma_0\cap\partial\Omega_\varepsilon} = 0;$$

and the identity

$$\int_{\Omega_\varepsilon} \left[u_t^{(1)}v_t - u_\rho^{(1)}v_\rho - \frac{1}{\rho^2}u_\varphi^{(1)}v_\varphi\right]\rho\, d\rho\, d\varphi\, dt = \int_{\Sigma_0\cap\partial\Omega_\varepsilon} \rho\alpha(\rho)u^{(1)}v\, d\rho\, d\varphi \tag{5.3}$$

holds for all $v \in V_\varepsilon$. We will show that the Fourier expansion

$$u^{(1)}(\rho,\varphi,t) = \sum_{n=0}^{\infty}\left\{u_n^{(11)}(\rho,t)\cos n\varphi + u_n^{(12)}(\rho,t)\sin n\varphi\right\} \tag{5.4}$$

has zero Fourier–coefficients $u_n^{(1i)}(\varrho,t)$ in Ω_ε, i.e. $u^{(1)} \equiv 0$ in Ω_ε.

Since $u^{(1)} \in C^1(\bar{\Omega}_\varepsilon)$, using

$$v_1(\rho,\varphi,t) = w(\rho,t)\cos n\varphi \in V_\varepsilon \quad \text{or} \quad v_2(\rho,\varphi,t) = w(\rho,t)\sin n\varphi \in V_\varepsilon$$

in (5.3), we derive

$$\int_{G_\varepsilon}\left[u^{(1i)}_{n,t}w_t - u^{(1i)}_{n,\rho}w_\rho - \frac{n^2}{\rho^2}u^{(1i)}_n w\right]\rho\, d\rho\, dt - \int_{\partial G_\varepsilon\cap S_0}\rho\alpha(\rho)u^{(1i)}_n w\, d\rho = 0 \tag{5.5}$$

for all $w \in V^{(1)}_\varepsilon, n \in \mathbb{N}, i = 1,2$. From Definition 2.2 it follows that the functions $u^{(1i)}_n(\varrho,t)$ are generalized solutions of the homogeneous problem $P_{\alpha,1}$ with $d(\varrho,t) = n^2\rho^{-2} \in C^\infty(\bar{G}_0\backslash(0,0))$. Clearly Theorem 4.2 gives $u^{(1i)}_n(\varrho,t) \equiv 0$ in Ω_ε for $n \in \mathbb{N}$, $i = 1,2$ and thus $u^{(1)} = u_1 - u_2 \equiv 0$ in Ω_ε.

Case $\varepsilon = 0$. In this case from Lemma 2.1 it follows that the generalized solution $u^{(1)} \in C^1(\bar{\Omega}_0 \setminus (0,0,0))$ of Problem P_α in Ω_0 is also a generalized solution of the homogeneous problem P_α in Ω_ε for each $\varepsilon \in (0,1)$. From Case 1 we know that $u^{(1)} \equiv 0$ in Ω_ε for each $\varepsilon > 0$ and thus $u^{(1)} = u_1 - u_2 \equiv 0$ in Ω_0. ■

THEOREM 5.2. *Let the function* $f \in C(\bar{\Omega}_0) \cap C^1(\bar{\Omega}_0\backslash(0,0,0))$ *be of the form:*

$$f^{(1)}(\varrho,\varphi,t) = \sum_{n=0}^{k}\left\{f^{(11)}_n(\varrho,t)\cos n\varphi + f^{(12)}_n(\varrho,t)\sin n\varphi\right\}. \tag{5.6}$$

Then there exists one and only one generalized solution

$$u^{(1)}(\varrho,\varphi,t) = \sum_{n=0}^{k}\left\{u^{(11)}_n(\varrho,t)\cos n\varphi + u^{(12)}_n(\varrho,t)\sin n\varphi\right\} \tag{5.7}$$

of the problem P_α *in* Ω_0, $u^{(1)} \in C^2(\bar{\Omega}_0 \setminus (0,.,0))$ *and it is a classical solution of the problem* P_α *in each domain* $\Omega_\varepsilon, \varepsilon \in (0,1)$. *Moreover, for a fixed n the corresponding trigonometric polynomial* u_n *of degree n satisfies a priori estimates: for* $n = 0$:

$$\begin{aligned}\|u_0(x_1,x_2,t)\|_{C^1(\tilde{\Omega}_\varepsilon)} &= \textstyle\sum_{|\alpha|\le 1}\sup_{\tilde{\Omega}_\varepsilon}|D^\alpha u_0|\cdot \\ &\le 8\exp(2M_\alpha)\varepsilon^{1/2}\exp(1/4\varepsilon^2)\|f^{(11)}_0\|_{C^0(\bar{G}_0)};\end{aligned} \tag{5.8}$$

for $n \in \mathbb{N}$:

$$\begin{aligned}&\|u_n(x_1,x_2,t)\|_{C^1(\tilde{\Omega}_\varepsilon)} \\ &\le 8\exp(2M_\alpha)\frac{\varepsilon^{1/2}}{n}\exp\left(\frac{n^2}{\varepsilon^2}\right)\left(\|f^{(11)}_n\|_{C^0(\bar{G}_0)} + \|f^{(12)}_n\|_{C^0(\bar{G}_0)}\right),\end{aligned} \tag{5.9}$$

where $\tilde{\Omega}_\varepsilon = \Omega_0 \cap \{(\varrho,t) : \varrho + t > \varepsilon\}$.

Proof. It is enough to consider the case of a fixed number n. Let

$$U^{(1)}(\varrho,t) = \begin{cases} u^{(11)}_n(\varrho,t) \text{ in case } F^{(1)}(\varrho,t) = f^{(11)}_n(\varrho,t), \\ u^{(12)}_n(\varrho,t) \text{ in case } F^{(1)}(\varrho,t) = f^{(12)}_n(\varrho,t). \end{cases} \tag{5.10}$$

Then by (5.7) and (5.10), the equation (5.1) becomes

$$\frac{1}{\varrho}(\varrho U^{(1)}_\varrho)_\varrho - U^{(1)}_{tt} - \frac{n^2}{\varrho^2}U^{(1)} = F^{(1)}(\varrho,t) \tag{5.11}$$

As in Section 2, we make the substitutions

$$\xi = 1 - \varrho - t\ ,\ \eta = 1 - \varrho + t, \tag{5.12}$$

and introduce the new function

$$U^{(2)}(\xi,\eta) = \varrho^{1/2} U^{(1)}(\varrho(\xi,\eta), t(\xi,\eta)). \tag{5.13}$$

Then (5.11) reduces to (2.18), where

$$\begin{aligned} c(\xi,\eta) &= \frac{1-4n^2}{4(2-\eta-\xi)^2} \in C^\infty(\bar{D}_0 \setminus (1,1))\ , \\ g(\xi,\eta) &= \frac{1}{4\sqrt{2}}(2-\eta-\xi)^{1/2} f_n^{(2i)}(\xi,\eta) \in C^1(\bar{D}_0 \setminus (1,1))\ , \\ f_n^{(2i)}(\xi,\eta) &= f_n^{(1i)}(\varrho(\xi,\eta), t(\xi,\eta)), \end{aligned} \tag{5.14}$$

and satisfies the Goursat–Darboux problem $P_{\alpha,2}$. Theorems 3.1 and 3.2 ensure the existence of a classical solution $U^{(2)} = U^{(2)}(\xi,\eta)$ of this problem with the properties (3.6).

Case $n \in \mathbb{N}$. In view of (3.5), (5.14), it is easy to see that

$$\begin{aligned} c(\varepsilon) &:= \sup_{D_\varepsilon^{(1)}} |c(\xi,\eta)| \le \frac{n^2}{\varepsilon^2}\ , \\ M_g &:= \sup_{D_\varepsilon^{(1)}} |\frac{1}{4\sqrt{2}}(2-\eta-\xi)^{1/2} f_n^{(2i)}(\xi,\eta)| \le \frac{1}{4} \|f_n^{(1i)}\|_{C^0(\bar{G}_0)}\ . \end{aligned} \tag{5.15}$$

where $D_\varepsilon^{(1)} = \{(\xi,\eta) | 0 < \xi < \eta < 1,\ 0 < \xi < 1-\varepsilon\}$, $\varepsilon > 0$. Hence Theorems 3.1 and 3.2, on one hand, ensure the smoothness of the solution $U^{(2)}$ of Problem $P_{\alpha,2}$, i.e.

$$U_n^{(2i)}(\xi,\eta) := U^{(2)} \in C^2(\bar{D}_\varepsilon^{(1)}), \tag{5.16}$$

on the other hand, they ensure the a priori estimates:

$$\begin{aligned} \sup_{D_\varepsilon^{(1)}} |U_n^{(2i)}(\xi,\eta)| &\le \frac{1}{4}\|f_n^{(1i)}\|_{C^\circ(\bar{G}_0)} \frac{\varepsilon^2}{n^2} \exp(M_\alpha) \exp\left(\frac{n^2}{\varepsilon^2}\right)\ , \\ \sup_{D_\varepsilon^{(1)}} \{|U_{n,\xi}^{(2i)}|,\ |U_{n,\eta}^{(2i)}|\} &\le \frac{1}{4}\|f_n^{(1i)}\|_{C^\circ(\bar{G}_0)} \frac{\varepsilon^2}{n^2} \exp(2M_\alpha) \exp\left(\frac{n^2}{\varepsilon^2}\right)\ . \end{aligned} \tag{5.17}$$

Also, by (5.12) and (5.13), we have

$$U_n^{(1i)}(\varrho,t) = \varrho^{-\frac{1}{2}} U_n^{(2i)}(\xi,\eta)\ .$$

Since $\varrho \ge \varepsilon/2$ for $(\xi,\eta) \in D_\varepsilon^{(1)}$, by the inverse transformation:

$$\begin{aligned} |u_n^{(1i)}(\varrho,t)| &\le \exp(M_\alpha)\frac{\varepsilon^{3/2}}{n^2} \exp\left(\frac{n^2}{\varepsilon^2}\right) \|f_n^{(1i)}\|_{C^\circ(\bar{G}_0)}\ , \\ |u_{n,t}^{(1i)}(\varrho,t)| &\le \exp(2M_\alpha)\frac{\varepsilon^{3/2}}{n^2} \exp\left(\frac{n^2}{\varepsilon^2}\right) \|f_n^{(1i)}\|_{C^\circ(\bar{G}_0)}\ , \\ |u_{n,\varrho}^{(1i)}(\varrho,t)| &\le 2\exp(2M_\alpha)\frac{\varepsilon^{1/2}}{n^2} \exp\left(\frac{n^2}{\varepsilon^2}\right) \|f_n^{(1i)}\|_{C^\circ(\bar{G}_0)}\ . \end{aligned} \tag{5.18}$$

Therefore, in view of (5.7) and (5.18), we derive

$$\begin{aligned}(5.19)\qquad &\|\frac{1}{\varrho}u^{(1)}_{n,\varphi}(\varrho,\varphi,t)\|_{C^\circ(\bar{\Omega}_\varepsilon)}\\ &\le \exp(2M_\alpha)\frac{\varepsilon^{1/2}}{n}\exp\left(\frac{n^2}{\varepsilon^2}\right)\left(\|f^{(11)}_n\|_{C^\circ(\bar{G}_0)}+\|f^{(12)}_n\|_{C^\circ(\bar{G}_0)}\right).\end{aligned}$$

Since $u_n(\varrho\cos\varphi,\varrho\sin\varphi,t)=u^{(1)}_n(\varrho,\varphi,t)$, obviously

$$|u_{n,x_i}(x_1,x_2,t)|\le 3\exp(2M_\alpha)\frac{\varepsilon^{1/2}}{n}\exp\left(\frac{n^2}{\varepsilon^2}\right)\left(\|f^{(11)}_n\|_{C^\circ(\bar{G}_0)}+\|f^{(12)}_n\|_{C^\circ(\bar{G}_0)}\right),$$

$i=1,2$. So, the estimate (5.9) holds in $\tilde{\Omega}_\varepsilon$.

Case $n=0$. In this case, by (5.6) and (5.7), it follows that $f^{(1)}_0(\varrho,\varphi,t)=f^{(11)}_0(\varrho,t)$ and $u_0(x_1,x_2,t)=u^{(1)}_0(\varrho,\varphi,t)=u^{(11)}_0(\varrho,t)$. Problem $P_{\alpha,2}$ in this case becomes

$$U^{(2)}_{\xi\eta}+c(\xi,\eta)U^{(2)}=g(\xi,\eta)\ ,\ U^{(2)}|_{\xi=0}=0\ ,\ U^{(2)}|_{\eta=\xi}=0,$$

where

$$c(\xi,\eta)=[2(2-\eta-\xi)]^{-2}\in C^\infty(\bar{D}_0\setminus\{1,1\})$$

and

$$c(\varepsilon)=\sup_{D^{(1)}_\varepsilon}|c(\xi,\eta)|\le\frac{1}{4\varepsilon^2}\ ,\ M_g\le\frac{1}{4}\|f^{(11)}_0\|_{C^\circ(\bar{G}_0)}$$

Arguments similar to the previous case lead to (5.8).■

The following theorem is an immediate consequence of Theorems 5.1 and 5.2

THEOREM 5.3. *Let the function* $f\in C^1(\bar{\Omega}_0)$ *be of the form*

$$f(\rho,\varphi,t)=\sum_{n=0}^{\infty}\{f^{(1)}_n(\rho,t)\cos n\varphi+f^{(2)}_n(\rho,t)\sin n\varphi\}. \tag{5.20}$$

Suppose that the Fourier coefficients $f^{(1)}_n(\rho,t)$ *and* $f^{(2)}_n(\rho,t)$ *satisfy*

$$\begin{aligned}(5.21)\qquad &\|f\|_{\exp(\varepsilon)}:=\exp\left(\frac{1}{4\varepsilon^2}\right)\|f^{(11)}_0\|_{C^\circ(\bar{G}_0)}\\ &+\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(\frac{n^2}{\varepsilon^2}\right)\left(\|f^{(11)}_n\|_{C^\circ(\bar{G}_0)}+\|f^{(12)}_n\|_{C^\circ(\bar{G}_0)}\right)<\infty.\end{aligned}$$

Then there exist one and only one generalized solution $u\in C^1(\tilde{\Omega}_\varepsilon)$ *of the problem* P_α *in* Ω_ε *and the a priori estimate*

$$\|u\|_{C^1(\tilde{\Omega}_\varepsilon)}\le 8\exp(2M_\alpha)\|f\|_{\exp(\varepsilon)} \tag{5.22}$$

holds. If the series (5.20) *is finite, then* $u\in C^2(\bar{\Omega}_0\setminus(0,0,0))$ *and it is a classical solution of the problem* P_α *in* $\Omega_\varepsilon,\varepsilon\in(0,1)$

REMARK 5.1. *Condition* (5.21) *is valid for each* $\varepsilon \in (0,1)$ *if there exists a function* ψ *with* $\psi(n) \to \infty$ *as* $n \to \infty$, *such that*

$$\sum_{n=1}^{\infty} \frac{1}{n} \exp(n^2\psi(n)) \left(\|f_n^{(11)}\|_{C^0(\bar{G}_0)} + \|f_n^{(12)}\|_{C^0(\bar{G}_0)} \right) < \infty. \tag{5.23}$$

REMARK 5.2. *As we see, the norm* (5.21) *on the right-hand side of* (5.22) *tends to infinity as* $\varepsilon \to 0$. *At this point, it is reasonable to remained that, according to Theorem 6.1 (see, the discussion in Introduction) the estimate* (5.22) *is satisfied also by the generalized solutions which have singularities at the point* $(0,0,0)$. *Therefore, the left-hand side of* (5.22) *tends to infinity as* $\rho \to 0$. *The above phenomenon is subject to the new paper* [9].

6. On the singularity of solutions of Problem P_α

For the wave equation

$$\Box u = \frac{1}{\varrho}(\varrho u_\varrho)_\varrho + \frac{1}{\varrho^2} u_{\varphi\varphi} - u_{tt} = f(\varrho, \varphi, t) \tag{6.1}$$

we consider again the boundary value problem P_α, i.e.

$$P_\alpha : \qquad \Box u = f \text{ in } \Omega_0, \qquad u|_{\Sigma_1} = 0, \qquad [u_t + \alpha(\varrho)u]|_{\Sigma_0} = 0. \tag{6.2}$$

and begin with the following interesting result of this section

THEOREM 6.1. *Let* $\alpha(\varrho) \geq 0$, $\varrho \in [0,1]$; $\alpha(\varrho) \in C^\infty([0,1])$. *Then for each* $n \in \mathbb{N}$, $n \geq 4$, *there exists a function* $f_n(\varrho, \varphi, t) \in C^{n-2}(\bar{\Omega}_0)$, *for which the corresponding general solution* u_n *of the problem* P_α *belongs to* $C^n(\bar{\Omega}_0 \backslash (0,0,0))$ *and the estimate*

$$|u_n(\rho, \varphi, \rho)| \geq \frac{1}{2}|u_n(2\rho, \varphi, 0)| + \rho^{-n}|\cos n\varphi| \geq \rho^{-n}|\cos n\varphi|, \qquad 0 < \rho < 1, \tag{6.3}$$

holds. In the case $\alpha(\varrho) \equiv 0$ *the upper estimate*

$$|u_n(\varrho, \varphi, t)| \leq c_\mu \rho^{-1/2} \left(\frac{\rho}{(\rho + t)(\rho - t)} \right)^{n - \frac{1}{2}} |\cos n\varphi|, (\varrho, t) \in D_1^\mu \tag{6.4}$$

holds, where $c_\mu = const$ *and*

$$D_1^\mu := \{(\rho, t) : 0 < \rho - t \leq \rho + t \leq \mu(\rho - t)\}, \mu < 2^{\frac{2n+1}{2n-1}} - 1.$$

Thus, for $\alpha(\varrho) \equiv 0$ *we have two-sided estimates, which in special cases* $t = \rho$ *and* $t = 0$ *are:*

$$\rho^{-n}|\cos n\varphi| \leq |u_n(\rho, \varphi, \rho)|, \qquad |u_n(\rho, \varphi, 0)| \leq C_2 \rho^{-n}|\cos n\varphi|, \tag{6.5}$$

with $C_2 = const$. *That is, in the case of Problem P2 the exact behavior of* $u_n(x_1, x_2, t)$ *around (0,0,0) is* $(x_1^2 + x_2^2)^{-n/2}$.

Proof. Note that, by Theorem 1.1, the functions

$$w_n(\varrho, \varphi, t) = \varrho^{-n}(\varrho^2 - t^2)^{n-1/2}(a_n \cos n\varphi + b_n \sin n\varphi), n \geq 4,$$

are classical solutions of Problem P_α^* with $\alpha \equiv 0$,.where obviously $w_n \in C^{n-2}(\bar{\Omega}_0)$.

We consider the special case of Problem P_α :

$$\Box u = \varrho^{-n}(\varrho^2 - t^2)^{n-1/2} \cos n\varphi \qquad \text{in } \Omega_0, \tag{6.6}$$

$$u|_{\Sigma_1} = 0, \qquad [u_t + \alpha(\varrho)u]|_{\Sigma_0} = 0. \tag{6.7}$$

The Theorem 5.1 declares that the problem (6.6), (6.7) has at most one generalized solution. On the other hand, from Theorem 5.2 we know that for this right-hand side there exists a generalized solution in Ω_0 of the form

$$u_n(\varrho,\varphi,t) = u_n^{(1)}(\varrho,t)\cos n\varphi \in C^{n-1}(\bar{\Omega}_0\backslash(0,0,0)),$$

which is classical solution in Ω_ε, $\varepsilon \in (0,1)$. By setting $u_n^{(2)}(\varrho,t) = \varrho^{\frac{1}{2}}u_n^{(1)}(\varrho,t)$ and substituting

$$\xi = 1-\varrho-t, \qquad \eta = 1-\varrho+t, \tag{6.8}$$

the problem (6.6), (6.7), in view of

$$U_n(\xi,\eta) = u_n^{(2)}(\varrho(\xi,\eta),t(\xi,\eta)), \tag{6.9}$$

becomes a Goursat–Darboux problem $P_{\alpha,2}$:

$$U_{n,\xi\eta} + c(\xi,\eta)U_n = g(\xi,\eta), \tag{6.10}$$

$$U_n(0,\eta) = 0, \qquad [U_{n,\eta} - U_{n,\xi} + \alpha(1-\xi)U_n]\big|_{\eta=\xi} = 0. \tag{6.11}$$

The coefficients

$$c(\xi,\eta) = \frac{1-4n^2}{4(2-\eta-\xi)^2} \in C^\infty(\bar{D}_\varepsilon^{(1)}), \quad n \geq 4, \tag{6.12}$$

$$g(\xi,\eta) = 2^{n-\frac{5}{2}}\left[\frac{(1-\xi)(1-\eta)}{2-\eta-\xi}\right]^{n-\frac{1}{2}} \in C^{n-1}(\bar{D}_\varepsilon^{(1)}) \tag{6.13}$$

are defined by (3.1). It is obvious that in this case $c(\xi,\eta) \leq 0$, $g(\xi,\eta) \geq 0$ in $\bar{D}_\varepsilon^{(1)}$, $\varepsilon \in (0,1)$.

Thus, for $\alpha(\xi) \geq 0$, in view of Theorem 3.1 and Lemma 3.1,we have the following result.

Proposition 6.1. *There exists a classical solution $U(\xi,\eta) \in C^n(\bar{D}_0 \setminus (1,1))$ for the problem* (6.10), (6.11) *for which*

$$U(\xi,\eta) \geq 0, \quad U_\xi(\xi,\eta) \geq 0, \quad U_\eta(\xi,\eta) \geq 0 \text{ in } \bar{D}_\varepsilon^{(1)}.$$

Let

$$K = \int_{D_{\frac{1}{2}}^{(1)}} g^2(\xi,\eta)\,d\eta\,d\xi > 0. \tag{6.14}$$

From (6.10) for $0 < \varepsilon < 1/2$ it follows that

$$\begin{aligned} 0 < K \leq \int_{D_\varepsilon^{(1)}} g^2(\xi,\eta)\,d\eta\,d\xi &= \int_{D_\varepsilon^{(1)}} U_{\xi\eta}g(\xi,\eta)\,d\eta\,d\xi \\ &+ \int_{D_\varepsilon^{(1)}} c(\xi,\eta)U(\xi,\eta)g(\xi,\eta)\,d\eta\,d\xi =: I_1 + I_2, \end{aligned} \tag{6.15}$$

where

$$
\begin{aligned}
I_1 &= \int_0^{1-\varepsilon}\int_\xi^1 (U_{\xi\eta}g)(\xi,\eta)\,d\eta\,d\xi \\
&= \int_0^{1-\varepsilon}[U_\xi(\xi,1)g(\xi,1) - U_\xi(\xi,\xi)g(\xi,\xi)]\,d\xi - \int_{D_\varepsilon^{(1)}}(U_\xi g_\eta)(\xi,\eta)\,d\eta\,d\xi.
\end{aligned}
$$

By (6.13), it is obvious that $g(\xi,1)=0$. So,

$$
I_1 = -\int_0^{1-\varepsilon} U_\xi(\xi,\xi)g(\xi,\xi)\,d\xi - \int_{D_\varepsilon^{(1)}}(U_\xi g_\eta)(\xi,\eta)\,d\eta\,d\xi. \tag{6.16}
$$

Since

$$
\begin{aligned}
&\int_{D_\varepsilon^{(1)}}(U_\xi g_\eta)(\xi,\eta)\,d\xi\,d\eta = \int_0^{1-\varepsilon}\int_0^\eta (U_\xi g_\eta)(\xi,\eta)\,d\xi\,d\eta \\
&+\int_{1-\varepsilon}^1\int_0^{1-\varepsilon}(U_\xi g_\eta)(\xi,\eta)\,d\xi\,d\eta = \int_0^{1-\varepsilon}[(Ug_\eta)(\eta,\eta)-(Ug_\eta)(0,\eta)]\,d\eta \\
&+\int_{1-\varepsilon}^1[(Ug_\eta)(1-\varepsilon,\eta)-(Ug_\eta)(0,\eta)]\,d\eta - \int_{D_\varepsilon^{(1)}}Ug_{\xi\eta}(\xi,\eta)\,d\xi\,d\eta \\
&= \int_0^{1-\varepsilon}(Ug_\eta)(\eta,\eta)\,d\eta + \int_{1-\varepsilon}^1 (Ug_\eta)(1-\varepsilon,\eta)\,d\eta \\
&- \int_{D_\varepsilon^{(1)}}(Ug_{\xi\eta})(\xi,\eta)\,d\xi\,d\eta,
\end{aligned} \tag{6.17}
$$

(6.16) becomes

$$
\begin{aligned}
I_1 = &-\int_0^{1-\varepsilon}[U_\xi(\xi,\xi)g(\xi,\xi)+U(\xi,\xi)g_\eta(\xi,\xi)]\,d\xi \\
&-\int_{1-\varepsilon}^1 U(1-\varepsilon,\eta)g_\eta(1-\varepsilon,\eta)\,d\eta + \int_{D_\varepsilon^{(1)}}(Ug_{\xi\eta})(\xi,\eta)\,d\xi\,d\eta.
\end{aligned} \tag{6.18}
$$

An elementary calculation shows that

$$
g_\xi(\xi,\eta) = -(n-\frac{1}{2})2^{n-\frac{5}{2}}\left[\frac{(1-\xi)(1-\eta)}{2-\eta-\xi}\right]^{n-\frac{3}{2}}\left[\frac{(1-\eta)}{2-\eta-\xi}\right]^2 \le 0, \tag{6.19}
$$

$$
g_\eta(\xi,\eta) = -(n-\frac{1}{2})2^{n-\frac{5}{2}}\left[\frac{(1-\xi)(1-\eta)}{2-\eta-\xi}\right]^{n-\frac{3}{2}}\left[\frac{(1-\xi)}{2-\eta-\xi}\right]^2 \le 0, \tag{6.20}
$$

and

$$
g_\xi(\xi,\xi) = g_\eta(\xi,\xi) = \frac{1}{16}(1-2n)(1-\xi)^{n-\frac{3}{2}}. \tag{6.21}
$$

From (6.18) and (6.15) it follows that

$$
\begin{aligned}
0 < K \le I_1 + I_2 = &- \int_0^{1-\varepsilon} [U_\xi(\xi,\xi)g(\xi,\xi) + U(\xi,\xi)g_\xi(\xi,\xi)]\, d\xi \\
&- \int_{1-\varepsilon}^1 U(1-\varepsilon,\eta)g_\eta(1-\varepsilon,\eta)\, d\eta + \int_{D_\varepsilon^{(1)}} U[g_{\xi\eta} + cg](\xi,\eta)\, d\xi\, d\eta.
\end{aligned} \tag{6.22}
$$

Also, it is easy to check that

$$
g_{\xi\eta}(\xi,\eta) + c(\xi,\eta)g(\xi,\eta) = 0.
$$

Thus,

$$
\begin{aligned}
0 < K \le I_1 + I_2 = &- \int_0^{1-\varepsilon} [U_\xi(\xi,\xi)g(\xi,\xi) + U(\xi,\xi)g_\xi(\xi,\xi)]\, d\xi \\
&- \int_{1-\varepsilon}^1 U(1-\varepsilon,\eta)g_\eta(1-\varepsilon,\eta)\, d\eta,
\end{aligned} \tag{6.23}
$$

where, as it is easy to check,

$$
g_\xi(\xi,\xi) = \frac{1}{2}[g(\xi,\xi)]_\xi. \tag{6.24}
$$

The function $U(\xi,\eta)$ is a classical solution of (6.10), (6.11) in $\bar{D}_\varepsilon$, $\varepsilon \in (0,1)$ with

$$
U_\xi(\xi,\xi) = \frac{1}{2}[U(\xi,\xi)]_\xi + \frac{1}{2}\alpha(1-\xi)U(\xi,\xi) \tag{6.25}
$$

If we substitute (6.24) and (6.25) into (6.23), we get

$$
\begin{aligned}
K \le I_1 + I_2 = &-\frac{1}{2}\int_0^{1-\varepsilon} [g(\xi,\xi)U(\xi,\xi)]_\xi\, d\xi \\
&-\frac{1}{2}\int_0^{1-\varepsilon} \alpha(1-\xi)U(\xi,\xi)g(\xi,\xi)\, d\xi - \int_{1-\varepsilon}^1 U(1-\varepsilon,\eta)g_\eta(1-\varepsilon,\eta)\, d\eta \\
&= -\frac{1}{2}(gU)(1-\varepsilon,1-\varepsilon) - \frac{1}{2}\int_0^{1-\varepsilon} \alpha(1-\xi)U(\xi,\xi)g(\xi,\xi)\, d\xi \\
&- \int_{1-\varepsilon}^1 U(1-\varepsilon,\eta)g_\eta(1-\varepsilon,\eta)\, d\eta.
\end{aligned} \tag{6.26}
$$

According to Proposition 6.1 and the choice of right-hand side of (6.8), we have

$$
U(\xi,\eta) \ge 0, U_\eta(\xi,\eta) \ge 0, \alpha(\xi) \ge 0, g(\xi,\eta) \ge 0, g_\eta(\xi,\eta) \le 0 \text{ in } \bar{D}_\varepsilon^{(1)},
$$

which together with (6.26) implies

$$K \leq I_1 + I_2 \leq -\int_{1-\varepsilon}^{1} U(1-\varepsilon,\eta) g_\eta(1-\varepsilon,\eta)\,d\eta - \frac{1}{2}(gU)(1-\varepsilon,1-\varepsilon)$$

$$= \int_{1-\varepsilon}^{1} U(1-\varepsilon,\eta) |g_\eta(1-\varepsilon,\eta)|\,d\eta - \frac{1}{2}(gU)(1-\varepsilon,1-\varepsilon)$$

$$\leq \int_{1-\varepsilon}^{1} U(1-\varepsilon,1) |g_\eta(1-\varepsilon,\eta)|\,d\eta - \frac{1}{2}(gU)(1-\varepsilon,1-\varepsilon)$$

$$= \left[U(1-\varepsilon,1) - \frac{1}{2}U(1-\varepsilon,1-\varepsilon)\right] g(1-\varepsilon,1-\varepsilon),$$

because $g(1-\varepsilon,1)=0$. Since $g(1-\varepsilon,1-\varepsilon) = \frac{1}{4}\varepsilon^{n-\frac{1}{2}}$, we see that

$$0 < K \leq \left[U(1-\varepsilon,1) - \frac{1}{2}U(1-\varepsilon,1-\varepsilon)\right] \frac{1}{4}\varepsilon^{n-\frac{1}{2}}.$$

For $\xi = 1-\varepsilon$, $\eta = 1$ we have $\varrho = t = \varepsilon/2$ and so

$$0 < 4K\varepsilon^{\frac{1}{2}-n} \leq u_n^{(2)}\left(\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right) - \frac{1}{2}u_n^{(2)}(\varepsilon,0). \tag{6.27}$$

Finally, the inverse transformation gives

$$u_n^{(1)}\left(\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right) \geq \frac{1}{2}u_n^{(1)}(\varepsilon,0) + \tilde{C}_1\varepsilon^{-n} \geq \tilde{C}_1\varepsilon^{-n},\ 0<\varepsilon<\frac{1}{2},$$

with $\tilde{C}_1 = 2^{\frac{5}{2}}K$. Multiplying the function u_n by $\tilde{C}_1^{-1}$, we see that (6.3) holds.

In order to obtain an upper estimate of the singular solution, we consider the case $\alpha(\varrho)\equiv 0$. In this case (6.26) gives

$$I_1 + I_2 = \int_{\bar{D}_\varepsilon^{(1)}} g^2(\xi,\eta)\,d\xi\,d\eta = -\frac{1}{2}(Ug)(1-\varepsilon,1-\varepsilon) - \int_{1-\varepsilon}^{1} (Ug_\eta)(1-\varepsilon,\eta)\,d\eta$$

Put

$$K_1 = \int_{D_0^{(1)}} g^2(\xi,\eta)\,d\xi\,d\eta > 0.$$

Then for $0<\delta<\varepsilon<1$ we have

$$\begin{aligned}
K_1 &\geq I_1 + I_2 \\
&= -\frac{1}{2}(gU)(1-\varepsilon, 1-\varepsilon) + \int_{1-\varepsilon}^{1} U(1-\varepsilon, \eta)|g_\eta(1-\varepsilon, \eta)|\, d\eta \\
&\geq -\frac{1}{2}(gU)(1-\varepsilon, 1-\varepsilon) + \int_{1-\delta}^{1} U(1-\varepsilon, 1-\varepsilon)|g_\eta(1-\varepsilon, \eta)|\, d\eta \\
&\geq -\frac{1}{2}(gU)(1-\varepsilon, 1-\varepsilon) + \int_{1-\delta}^{1} U(1-\varepsilon, 1-\delta)|g_\eta(1-\varepsilon, \eta)|\, d\eta \\
&\geq -\frac{1}{2}(gU)(1-\varepsilon, 1-\varepsilon) + (gU)(1-\varepsilon, 1-\delta) \\
&\geq U(1-\varepsilon, 1-\delta)\left[g(1-\varepsilon, 1-\delta) - \frac{1}{2}g(1-\varepsilon, 1-\varepsilon)\right] \\
&\geq \lambda(gU)(1-\varepsilon, 1-\delta),
\end{aligned} \tag{6.28}$$

where the constant $\lambda > 0$ is such that

$$(1-\lambda)g(1-\varepsilon, 1-\delta) \geq g(1-\varepsilon, 1-\varepsilon). \tag{6.29}$$

Using the explicit formula (6.16) for the function $g(\xi, \eta)$, we see that the last inequality is equivalent to

$$(1-\lambda)\left(\frac{\delta}{\varepsilon+\delta}\right)^{n-\frac{1}{2}} \geq 2^{-n+\frac{1}{2}}, \tag{6.30}$$

which implies

$$0 < \lambda \leq 1 - \frac{1}{2}\left(\frac{\varepsilon+\delta}{2\delta}\right)^{n-\frac{1}{2}}. \tag{6.31}$$

A necessary condition, for (6.31) to be satisfied is

$$1 \leq \frac{\varepsilon}{\delta} < 2^{\frac{2n+1}{2n-1}} - 1. \tag{6.32}$$

Using (6.32), we can find an upper estimate for the generalized solution u_n in this concrete case. To do that we consider the domain

$$D^\mu := \{(\xi, \eta) : 1-\eta \leq 1-\xi \leq \mu(1-\eta)\}, \tag{6.33}$$

where $1 \leq \mu < 2^{\frac{2n+1}{2n-1}} - 1$. Observe that

$$\inf_{D^\mu}\left\{1 - \frac{1}{2}\left(\frac{1-\xi+1-\eta}{2(1-\eta)}\right)^{n-\frac{1}{2}}\right\} = 1 - \frac{1}{2}\left(\frac{1+\mu}{2}\right)^{n-\frac{1}{2}} =: C_\mu > 0.$$

For $\lambda = C_\mu$, the inequalities (6.30) and (6.29) are satisfied and so, by (6.28), we see that

$$U(\xi, \eta) \leq 2^{-n+5/2} K_1 C_\mu^{-1}\left(\frac{2-\xi-\eta}{(1-\xi)(1-\eta)}\right)^{n-\frac{1}{2}}, (\xi, \eta) \in D^\mu. \tag{6.34}$$

By (6.9) and (6.8), the inequality (6.34) transforms to

$$u_n^{(2)}(\varrho, t) \leq 4K_1 C_\mu^{-1}\left(\frac{\rho}{(\rho+t)(\rho-t)}\right)^{n-\frac{1}{2}}, \tag{6.35}$$

which is satisfied for

$$(\varrho, t) \in D_1^\mu := \{0 < \rho - t \le \rho + t \le \mu(\rho - t)\}.$$

Finally, (6.35) implies

$$u_n^{(1)}(\varrho, t) \le 4K_1 C_\mu^{-1} \rho^{-1/2} \left(\frac{\rho}{(\rho + t)(\rho - t)} \right)^{n - \frac{1}{2}} \text{ for } (\varrho, t) \in D_1^\mu, \tag{6.36}$$

which coincides with the estimate (6.4)

Note that $C_\mu = 1/2$ on $\{t = 0\}$ and so

$$u_n^{(1)}(\rho, 0) \le 8K_1 \rho^{-n}, \qquad 0 < \rho < 1, \tag{6.37}$$

which is the upper estimate in (6.5). The proof of theorem is complete. ■

We conclude this section with

THEOREM 6.2. *Let $\alpha(\varrho) \ge 0$ for $\rho \in [0,1]$, $\alpha \in C^{n-2}[0,1]$. Then for $n \in \mathbb{N}$, $n \ge 4$ there exists a function $f_{n1}(\varrho, \varphi, t) \in C^{n-2}(\bar{\Omega}_0)$ (different from the function of Theorem 6.1) such that for the corresponding to it generalized solution u_n of the problem P_α*

$$u_n(\rho, \varphi, t) \in C^{n-1}(\bar{\Omega}_0 \setminus (0,0,0)),$$

$$u_n(\rho, \varphi, \rho) \ge u_n(2\rho, \varphi, 0) + \rho^{1-n} |\cos n\varphi| \ge \rho^{1-n} |\cos n\varphi|. \tag{6.38}$$

Proof. The functions

$$v_n(\rho, \varphi, t) = t\rho^{-n}(\rho^2 - t^2)^{n-3/2}(a_n \cos n\varphi + b_n \sin n\varphi)$$

are classical solutions of Protter's problem $P1^*$. We consider the problem

$$\Box u = t\rho^{-n}(\rho^2 - t^2)^{n-3/2} \cos n\varphi \tag{6.39}$$

$$u|_{\Sigma_1} = 0, \ [u_t + \alpha(\rho) u]|_{\Sigma_0} = 0. \tag{6.40}$$

According to Theorem 5.1, the problem (6.39), (6.40) has at most one generalized solution. Simultaneously Theorem 5.2 for this right-hand side ensure the existence of a generalized solution in Ω_0, which is of the form

$$u_n(\varrho, \varphi, t) = u_n^{(1)}(\varrho, t) \cos n\varphi \in C^{n-1}(\bar{\Omega}_0 \setminus (0,0,0))$$

and is a classical solution in Ω_ε, $\varepsilon \in (0,1)$.

Using the substitutions $u_n^{(2)}(\varrho, t) = \varrho^{\frac{1}{2}} u_n^{(1)}(\varrho, t)$, (6.8) and (6.9), the problem (6.39), (6.40) becomes a Goursat-Darboux problem

$$U_{n,\xi\eta} + c(\xi, \eta) U_n = g(\xi, \eta), \tag{6.41}$$

$$U_n(0, \eta) = 0, \qquad [U_{n,\eta} - U_{n,\xi} + \alpha(1 - \xi) U_n]|_{\eta = \xi} = 0, \tag{6.42}$$

where $c(\xi, \eta)$ is defined by (6.12), while

$$g(\xi, \eta) = 2^{n - \frac{7}{2}} (\eta - \xi)(2 - \eta - \xi)^{\frac{1}{2} - n} [(1 - \eta)(1 - \xi)]^{n - \frac{3}{2}} \in C^{n-2}(\bar{D}_\varepsilon^{(1)}). \tag{6.43}$$

From (6.10) and (6.43) it follows that $c(\xi, \eta) \le 0$, $g(\xi, \eta) \ge 0$ in $\bar{D}_\varepsilon^{(1)}$ for $\varepsilon \in (0,1)$. Hence Theorem 3.1 and Lemma 3.1 imply

Proposition 6.2. *There exists a classical solution* $U(\xi,\eta) \in C^{n-1}(\bar{D}_0 \setminus (1,1))$ *for the problem* (6.41), (6.42) *for which*

$$U(\xi,\eta) \geq 0, \quad U_\eta(\xi,\eta) \geq 0, \quad U_\xi(\xi,\eta) \geq 0 \quad in \ \bar{D}^{(1)}_\varepsilon.$$

A elementary calculation shows that $g(\xi,\xi) = 0$,

$$g_\eta(\xi,\xi) = -g_\xi(\xi,\xi) = \frac{1}{8}(1-\xi)^{n-\frac{5}{2}} \geq 0 \tag{6.44}$$

$$g_{\xi\eta}(\xi,\eta) + c(\xi,\eta)g(\xi,\eta) = 0. \tag{6.45}$$

Since

$$g_\eta(\xi,\eta) = g(\xi,\eta)\left[\frac{1}{\eta-\xi} + \frac{n-\frac{1}{2}}{2-\eta-\xi} - \frac{n-\frac{3}{2}}{1-\eta}\right]$$

and

$$g_\eta(1-\varepsilon,\eta) = \frac{\varepsilon g(1-\varepsilon,\eta)}{(1-\eta)(\varepsilon^2-(1-\eta)^2)}\left[\frac{1}{2} + n - \eta(\frac{1}{2}+n) + \varepsilon(\frac{3}{2}-n)\right],$$

for

$$\eta_\varepsilon = 1 - \varepsilon\frac{2n-3}{2n+1}$$

we have

$$g_\eta(1-\varepsilon,\eta) > 0 \text{ for } 1-\varepsilon < \eta < \eta_\varepsilon, \tag{6.46}$$

$$g_\eta(1-\varepsilon,\eta) < 0 \text{ for } \eta_\varepsilon < \eta < 1. \tag{6.47}$$

To show (6.38), let

$$K_2 = \int_{\bar{D}^{(1)}_{\frac{1}{2}}} g^2(\xi,\eta)\, d\xi\, d\eta > 0.$$

Then

$$0 < K_2 \leq \int_{\bar{D}^{(1)}_\varepsilon} g^2(\xi,\eta)\, d\xi\, d\eta, \qquad 0 < \varepsilon < \frac{1}{2}.$$

Using arguments similar to that of Theorem 6.1, we arrive to (6.18). By (6.45), we get

$$0 < K_2 \leq \int_{D^{(1)}_\varepsilon} g^2(\xi,\eta)\, d\xi\, d\eta = -\int_{1-\varepsilon}^{1} U(1-\varepsilon,\eta)g_\eta(1-\varepsilon,\eta)\, d\eta.$$
$$-\int_0^{1-\varepsilon} [U_\xi(\xi,\xi)g(\xi,\xi) + U(\xi,\xi)g_\eta(\xi,\xi)]\, d\xi$$

Since $g(\xi,\xi) = 0$, the above inequality becomes

$$0 < K_2 \leq -\int_0^{1-\varepsilon} U(\xi,\xi)g_\eta(\xi,\xi)\, d\xi - \int_{1-\varepsilon}^{\eta_\varepsilon} U(1-\varepsilon,\eta)g_\eta(1-\varepsilon,\eta)\, d\eta$$
$$-\int_{\eta_\varepsilon}^{1} U(1-\varepsilon,\eta)g_\eta(1-\varepsilon,\eta)\, d\eta.$$

Following the steps of the proof of Theorem 6.1 and using the Proposition 6.2, we find

$$\begin{aligned}
0 < K_2 &\le \int_{\eta_\varepsilon}^{1} U(1-\varepsilon,\eta)|g_\eta(1-\varepsilon,\eta)|\,d\eta - \int_{1-\varepsilon}^{\eta_\varepsilon} U(1-\varepsilon,\eta)|g_\eta(1-\varepsilon,\eta)|\,d\eta \\
&\le \int_{\eta_\varepsilon}^{1} U(1-\varepsilon,1)|g_\eta(1-\varepsilon,\eta)|\,d\eta - \int_{1-\varepsilon}^{\eta_\varepsilon} U(1-\varepsilon,1-\varepsilon)|g_\eta(1-\varepsilon,\eta)|\,d\eta \\
&= [U(1-\varepsilon,1) - U(1-\varepsilon,1-\varepsilon)]\,g(1-\varepsilon,\eta_\varepsilon).
\end{aligned}$$

By (6.43), it follows that

$$g(1-\varepsilon,\eta_\varepsilon) \le \varepsilon^{n-\frac{3}{2}}$$

and so

$$0 < K_2 \le [U(1-\varepsilon,1) - U(1-\varepsilon,1-\varepsilon)]\,\varepsilon^{n-\frac{3}{2}}.$$

Finally, using (6.9), it follows that

$$0 < K_2 \le \left[u_n^{(2)}\left(\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right) - u_n^{(2)}(\varepsilon,0)\right]\varepsilon^{n-\frac{3}{2}},$$

i.e.

$$u_n^{(1)}(\varrho,\varrho) \ge u_n^{(1)}(2\varrho,0) + K_2\varrho^{1-n} \ge K_3\varrho^{1-n}, K_3 = 2^{1-n}K_2,$$

and so the estimate (6.38) holds. The proof of Theorem 6.2 is complete.■

REMARK 6.1. *In* [**2**], *Theorem 2, Aldashev considers the following type problems:*

Find a solution of the homogeneous wave equation $\Box u = 0$ in Ω_0, satisfying *the nonhomogeneous boundary conditions:*

$$\begin{aligned}
&P1': \quad u|_{\Sigma_0} = \tau_0(x) \quad , \quad u|_{\Sigma_1} = \sigma_1(x) \qquad \text{or} \\
&P2': \quad u_t|_{\Sigma_0} = \nu_0(x) \quad , \quad u|_{\Sigma_1} = \sigma_1(x).
\end{aligned}$$

Under certain conditions, imposed on the functions τ_0, σ_1, ν_0, he asserts that both Problems $P1'$ and $P2'$ are solvable in the class $C(\bar{\Omega}_0) \cap C^2(\Omega_0)$.

Comparing these conclusion with Theorems 6.1, 6.2 and the results presented in [**21**], it is not difficult to see the appearing contradiction. Indeed, applying the Duhamel's formula to the nonhomogeous wave equation (6.6) in Ω_0 with homogeneous Cauchy initial dates on Σ_0, we find the solution of this problem in $C^{n-1}(\bar{\Omega}_0)$, expressed by explicit formulas (see, [**23**], pp. 226-234). Therefore, the problem (6.6), (6.7) transforms to the problem $P2'$ with $v(x) \equiv 0$ and $\sigma_1 \in C^{n-1}(\bar{\Sigma}_0)$. But the last problem cannot be solved in $C(\bar{\Omega}_0)$, because, by Theorem 6.1, for $\alpha \equiv 0$ the unique generalized solution of Problem P_α has a power-type singularity of the form ρ^{-n} (see, (6.3)) at the point $(0,0,0)$.

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